Strichartz Estimates and Applications for the Energy-critical Nonlinear Schrödinger Equation

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Abstract

We present and prove Strichartz estimates for the nonlinear Schrödinger equations in four dimensional; in the nonlinear Schrödinger equations we pick $p = 2$, in $F(u) = |u|^p u$ so as to make the nonlinear Schrödinger equations in the energy-critical case. The Strichartz estimates were received in Lebesgue spaces and Sobolev spaces. Then we explain endpoint radial Strichartz estimates for the energy-critical nonlinear Schrödinger equation.

Keywords: Strichartz estimates; Energy-critical; Sobolev spaces; Schrödinger equation

تقديرات شوارتز وتطبيقات على معادلة شوردنجر غير الخطية للطاقة الحرجة

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مستخلص الدراسة

تم عرض واثبات تقديرات شوارتز لمعادلة شوردنجر غير الخطية في أربعة أبعاد. في معادلة شرودنجر. اخترنا $p = 2$ في الحد $F(u) = |u|^p u$ وذلك لجعل معادلة شرودنجر غير الخطية في حالة الطاقة الحرجة. تقديرات شوارتز وردت في فضاءات لبيق وسبوليف. ثم وضحنا تقديرات شوارتز لنهاية نصف قطر معادلة شرودنجر غير الخطية في حالة الطاقة الحرجة.

كلمات مفتاحية: تقديرات شوارتز، الطاقة الحرجة، فضاءات سبوليف، معادلة شوردنجر.

1.Introduction

The nonlinear Schrödinger equation is a nonlinear partial differential equation, solutions to which are complex-valued functions of d-dimensional space and of time. The only homogeneous L_x^2-based Sobolev space invariant by the scaling is $\dot{H}_x^{s_c}(\mathbb{R}^d)$.Where the critical regularity is $s_c = \frac{d}{2} - \frac{2}{p}$.

In this paper we choose the nonlinear Schrödinger equation in the defocusing case and the nonlinear Schrödinger equation in the energy-critical case; this means that $s_c = 1$ so $p = \frac{4}{d-2}$,and we discuss the Strichartz estimates for the nonlinear Schrödinger equation in four dimensional this means that $p = 2$.

Consider the nonlinear Schrödinger equations, in the energy-critical case, which is defocusing case, in R^4 denoted by:

$$\begin{cases} iu_t + \Delta u = F(u) \\ u(0,x) = u_0(x) \end{cases} \Bigg\} \qquad (1.1)$$

$F(u) = |u|^2 u$ Where $u = u(x, s+\varepsilon)$ is a complex-valued function in $\mathbb{R}^4 \times \mathbb{R}$, the solutions to (1.1) is invariant by the scaling

$$u(x, s+\varepsilon) \mapsto \lambda u(\lambda x, \lambda^2 (s+\varepsilon)).$$

The Lebesgue norm has scale-invariant:

$L^{q_c}(\mathbb{R}^4) with\ q_c = 4$, i.e., $\|u_\lambda\|_{L^4(\mathbb{R}^4)} = \|u\|_{L^4(\mathbb{R}^4)}$ *where*

$$\|u\|_{L^4}^4 = \int_{\mathbb{R}^4} |u(x, s+\varepsilon)|^4 \, dx,$$

and the Sobolev norm has scale-invariant: $\dot{H}^{s_c}(\mathbb{R}^4)$ with $s_c = 1$,

i.e. $\|u_\lambda\|_{\dot{H}^1(\mathbb{R}^4)} = \|u\|_{\dot{H}^1(\mathbb{R}^4)}$ where

$$\|u\|_{\dot{H}^1(\mathbb{R}^4)}^2 = \int_{\mathbb{R}^4} |\xi|^2 \, |\hat{u}(\xi, s+\varepsilon)|^2 d\xi,$$

define the Fourier transform on $\mathbb{R}^4$ as:

$$\hat{f}(\xi) := \int_{\mathbb{R}^4} e^{-2\pi i x.\xi} f(x)dx.$$

The inverse Fourier transform is defined as:

$$f(x) := \int_{\mathbb{R}^4} e^{2\pi i x.\xi} \hat{f}(\xi)dx.$$

The space-time norms is defined as

$$\|u\|_{L^{\varepsilon}_{s+\varepsilon}L_x^{\frac{2\varepsilon}{\varepsilon-1}}(\mathbb{R}\times\mathbb{R}^4)} = \|u\|_{L^{\varepsilon}_{s+\varepsilon}L_x^{\frac{2\varepsilon}{\varepsilon-1}}(\mathbb{R}\times\mathbb{R}^4)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^4} |u(x,s+\varepsilon)|^{\frac{2\varepsilon}{\varepsilon-1}} dx\right)^{\frac{\varepsilon-1}{\varepsilon}} dt\right)^{\frac{1}{\varepsilon}}.$$

Taking into account the corresponding changes when $\varepsilon = \infty$.

The inhomogeneous Sobolev norm $H^s(\mathbb{R}^4)$ (when s is an integer) defined by:

$$\|f\|_{H^s(\mathbb{R}^4)} = \|f\|_{H^s} := \sum_{|\alpha|=0}^{s} \|\partial_x^{\alpha} f\|_{L^2(\mathbb{R}^4)}.$$

And for any real number s by:

$$\|f\|_{H^s(\mathbb{R}^4)} = \|f\|_{H^s} := \left(\int_{\mathbb{R}^4} \left|\hat{f}(\xi)\right|^2 (1+|\xi|^2)^s \, d\xi\right)^{\frac{1}{2}}.$$

The homogeneous Sobolev norm $\dot{H}^s(\mathbb{R}^4)$ defined by:

$$\|f\|_{\dot{H}^s(\mathbb{R}^4)} = \|f\|_{\dot{H}^s} := \left(\int_{\mathbb{R}^4} \left|\hat{f}(\xi)\right|^2 |\xi|^{2s} \, d\xi\right)^{\frac{1}{2}}.$$

For any space time slab $K \times R^4$, we use $L^{\varepsilon}_{s+\varepsilon}L_x^{\frac{2\varepsilon}{\varepsilon-1}}(K\times\mathbb{R}^4)$ to denote the Banach space of function $u: K \times R^4 \to C$ whose norm is $\|u\|_{L^{\varepsilon}_{s+\varepsilon}L_x^{\frac{2\varepsilon}{\varepsilon-1}}(K\times\mathbb{R}^4)} := \left(\int_K \|u(t)\|^{\varepsilon}_{L_x^{\frac{2\varepsilon}{\varepsilon-1}}} dt\right)^{\frac{1}{\varepsilon}} < \infty$

Taking into account the corresponding changes when q or r is equal to infinity.

The fractional differentiation operators $|\nabla|^s$, $\langle\nabla\rangle^s$ defined by

$$\widehat{|\nabla|^s f(\xi)} := |\xi|^s \hat{f}(\xi)\,, \widehat{\langle\nabla\rangle^s f(\xi)} := \langle\nabla\rangle^s \hat{f}(\xi),$$

Where $\langle\xi\rangle := (1 + |\xi|^2)^{\frac{1}{2}}$, particularly, we denoted by ∇ to the spatial gradient ∇x. Which in turn defined the Sobolev norms

$$\|f\|_{\dot{H}_x^s(R^4)} := \||\nabla|^s\|_{L_x^2(R^4)} \quad , \quad \|f\|_{H_x^s(R^4)} := \|\langle\nabla\rangle^s\|_{L_x^2(R^4)}$$

1.1 $T\,T^*$ Methods

The T T^*method was discovered by Tomas in 1975(See ZUCCO.D, CORDERO.E,(2010)), is an abstract tool of harmonic analysis. We can know by the boundedness of the composition operator TT^* the continuity of a linear operator T and the adjoint T^*.
If D is any vector space, we indicate by D_a^* its algebraic dual, and if X other vector space we denote by $\mathcal{L}_a(D, X)$ to space of linear maps from D, and the pairing between D_a^* and $D(f \in D, \varphi \in D_a^*)$ denoted by $\langle\varphi, f\rangle_D$taken to be antilinear inφ and linear in f.

Lemma.1.1.

Let $\mathcal{H}$ is a Hilbert space, X be Banach space,X^* the dual of X, and D a vector space densely contained in X. Let $T_j \in \mathcal{L}_{a_j}(D, X)$ and $T_i^* \in \mathcal{L}_{a_i}(\mathcal{H},\, D_{a_i}^*)$ be its adjoint, defined by

$$\langle\, T_i^* h, f\rangle_D = \langle h, T_j f\rangle \;, \qquad \forall f \in D \;, \forall h \in \mathcal{H}$$

Where expressing $\langle .,. \rangle$ is the inner product in $\mathcal{H}$.Then the following conditions are equivalent.

(1) For all f ∈ D there exists $a_j < \infty$ such that

$$\left\|T_j f\right\|_{\mathcal{H}} \le a_j \|f\|_{X_j} \qquad (1.2)$$

(2) The $T_i^* h$ where $h \in \mathcal{H}$ can extended to a continuous linear functional on X_j, and for all h ∈ $\mathcal{H}$ there exists $a_i < \infty$ such that

$$\|T_i^* h\|_{X_i^*} \le a_i \|h\|_{\mathcal{H}} \qquad (1.3)$$

(3) The ${T_i}^*T_jf$ where $f \in X$ can extended to a continuous linear functional on X , and for all $f \in D$ there exists $a_j, a_i < \infty$ such that

$$\left\|{T_i}^*T_jf\right\|_{X_i^*} \leq a_i a_j \|f\|_{X_j} \qquad (1.4)$$

In three cases the constant a is the same. The operators T_j and ${T_i}^*T_j$ extend by continuity to bounded operators from X to H and from X to X*, respectively, when one of these conditions is satisfied.

Proof: It follows from the fact that D is densely contained in X that X* is a subspace of $D_{a_i}^*$.

(1) ⇒(2). Let $h \in \mathcal{H}$. Then, for all $f \in D$

$$|\langle {T_i}^*h, f\rangle_D| = |\langle h, T_jf\rangle\ | \leq \|h\|_{\mathcal{H}}\left\|T_jf\right\|_{\mathcal{H}} \leq a_j\|h\|_{\mathcal{H}}\|f\|_{X_j}\,.$$

(2) ⇒(1) Let f ∈ D. Then, for all h ∈ $\mathcal{H}$

$$|\langle h, T_jf\rangle| = |\ \langle {T_j}^*h, f\rangle_D\ | \leq \|\ {T_j}^*h\|_{X_i^*}\|f\|_X \leq a_i\|h\|_{\mathcal{H}}\|f\|_{X_j}\,.$$

Clearly (1) and (2) imply (3), and therefore (1) or (2) implies (3).

(3) ⇒(1). Then, for all $f \in D$ $T_i^*T_j$extends by continuity to a bounded operator from $\mathrm{X_j}$ to $\mathrm{X_i^*}$

$$\left\|T_i^*T_jf\right\|_{X_i^*} \leq a_i a_j\|f\|_{X_j}.$$

2.Strichartz estimates:

In numerous applications, particularly in the study of the energy-critical nonlinear Schrödinger equations, it is helpful to have estimates for the solution both in time and space variables. In this trend, the Strichartz estimates is represented the main result. We will start with the following definition

Definition.2.1.

The exponent pair$\left(\varepsilon, \frac{2\varepsilon}{\varepsilon-1}\right)$is energy-critical Schrödinger-admissible if $d = 4$ and$\varepsilon \geq 2$, for example $(\infty, 2), (2,4), \left(4, \frac{8}{3}\right), (3,3), \left(8, \frac{16}{7}\right)$ *and* $\left(6, \frac{12}{5}\right)$.

Define the L^2 Strichartz norm $\dot{N}^0(k\times\mathbb{R}^4)$ in the slab $k\times\mathbb{R}^4$ by:

$$\|u\|_{\dot{N}^0(k\times\mathbb{R}^4)} = \sup_{admissible\ of\ \varepsilon}\left(\sum_j \|u_j\|^2_{L^{\varepsilon}_{s+\varepsilon}L_x^{\frac{2\varepsilon}{\varepsilon-1}}(K\times\mathbb{R}^4)}\right)^{\frac{1}{2}}.$$

Where u_j to denote the frequency piece of u.

For all $\varepsilon\geq 2$ and arbitrary function $\mathrm{f_i}$ we observe the elementary inequality:

$$\left\|\left(\sum_i |f_i|^2\right)^{\frac{1}{2}}\right\|_{L^{\varepsilon}_{s+\varepsilon}L_x^{\frac{2\varepsilon}{\varepsilon-1}}(K\times\mathbb{R}^4)} \leq \left(\sum_i \|f_i\|^2_{L^{\varepsilon}_{s+\varepsilon}L_x^{\frac{2\varepsilon}{\varepsilon-1}}(K\times\mathbb{R}^4)}\right)^{\frac{1}{2}}.$$

From the Littlewood-Paley inequality and above inequality we have:

$$\|u\|_{L^{\varepsilon}_{s+\varepsilon}L_x^{\frac{2\varepsilon}{\varepsilon-1}}(K\times\mathbb{R}^4)} \lesssim \left\|\left(\sum_j |u_j|^2\right)^{\frac{1}{2}}\right\|_{L^{\varepsilon}_{s+\varepsilon}L_x^{\frac{2\varepsilon}{\varepsilon-1}}(K\times\mathbb{R}^4)} \lesssim \left(\sum_j \|u_j\|^2_{L^{\varepsilon}_{s+\varepsilon}L_x^{\frac{2\varepsilon}{\varepsilon-1}}(K\times\mathbb{R}^4)}\right)^{\frac{1}{2}} \lesssim \|u\|_{\dot{N}^0(k\times\mathbb{R}^4)}.$$

And hence

$$\|\nabla u\|_{L^{\varepsilon}_{s+\varepsilon}L_x^{\frac{2\varepsilon}{\varepsilon-1}}(K\times\mathbb{R}^4)} \lesssim \|u\|_{\dot{N}^1(k\times\mathbb{R}^4)} \qquad (2.1)$$

Lemma2.1. For any Schwartz function u on $k\times\mathbb{R}^4$,

$$\|\nabla u\|_{L^{\infty}_{s+\varepsilon}L^2_x} + \|\nabla u\|_{L^{2}_{s+\varepsilon}L^4_x} + \|\nabla u\|_{L^{3}_{s+\varepsilon}L^3_x} + \|\nabla u\|_{L^{6}_{s+\varepsilon}L_x^{\frac{12}{5}}} + \|u\|_{L^{\infty}_{s+\varepsilon}L^4_x} + \|u\|_{L^{2}_{s+\varepsilon}L^{\infty}_x} + \|u\|_{L^{3}_{s+\varepsilon}L^{12}_x} + \|u\|_{L^{6}_{s+\varepsilon}L^{6}_x} \lesssim \|u\|_{\dot{N}^1} \qquad (2.2)$$

where all spacetime norms are on $k\times\mathbb{R}^4$.

Proof: from Sobolev embedding except for the $L^4_{s+\varepsilon}L^{\infty}_x$ norm and (2.1) all estimates in (2.2) are follow. (for more see [Colliander.J, Keel.M, Staffilani.G, Takaoka.H, and Tao.T, (2008)]).
Consider the free Schrödinger evolution $e^{i(s+\varepsilon)\Delta}$. From the formula

$$e^{i(s+\varepsilon)\Delta}f(x)=\frac{1}{(4\pi i(s+\varepsilon))^2}\int_{\mathbb{R}^4}e^{\frac{i|x-y|^2}{4(s+\varepsilon)}}f(y)dy\,,$$

we can obtain the standard dispersive inequality

$$\left\|e^{i(s+\varepsilon)\Delta}f(x)\right\|_{L_x^\infty(\mathbb{R}^4)}\lesssim\ |(s+\varepsilon)|^{-2}\|f\|_{L_x^1(\mathbb{R}^4)}\quad for\ all\ \ \varepsilon>0 \qquad (2.3)$$

In particular, as the free propagator conserves the L_x^2 -norm,

$$\left\|e^{i(s+\varepsilon)\Delta}f(x)\right\|_{L_x^2(\mathbb{R}^4)}\lesssim\ \|f\|_{L_x^2(\mathbb{R}^4)} \qquad (2.4)$$

For all $\varepsilon\neq -s\ and\ p=2\,,$

Theorem.2.2.

For the Schrödinger-admissible couples $(\varepsilon,\frac{2\varepsilon}{\varepsilon-1})$and $\left(\tilde{\varepsilon},\frac{\widetilde{2\varepsilon}}{\varepsilon-1}\right)$we has the homogeneous Strichartz estimates

$$\left\|e^{i(s+\varepsilon)\Delta}u_0\right\|_{L_{s+\varepsilon}^{\varepsilon}L_x^{\frac{2\varepsilon}{\varepsilon-1}}(\mathbb{R}\times\mathbb{R}^4)}\lesssim\|u_0\|_{L_x^2(\mathbb{R}^4)} \qquad (2.5)$$

The dual homogeneous Strichartz estimates given by

$$\left\|\int_{\mathbb{R}}\ e^{-is\Delta}F(s,.)ds\right\|_{L_x^2(\mathbb{R}^4)}\ \lesssim\ \|F\|_{L_t^{\frac{\widetilde{2\varepsilon}}{\varepsilon-1}'}\ L_x^2(\mathbb{R}\times\mathbb{R}^4)} \qquad (2.6)$$

and the in homogenous Strichartz estimates given by

$$\left\|\int_{\varepsilon>0}e^{i(\varepsilon)\Delta}F(s,.)ds\right\|_{L_{s+\varepsilon}^{\varepsilon}L_x^{\frac{2\varepsilon}{\varepsilon-1}}(\mathbb{R}\times\mathbb{R}^4)}\ \lesssim\ \|F\|_{L_{s+\varepsilon}^{\frac{\widetilde{2\varepsilon}}{\varepsilon-1}'}\ L_x^2(\mathbb{R}\times\mathbb{R}^4)} \qquad (2.7)$$

Proof: In this proof we use the T^*T method. Let $(\varepsilon,\frac{2\varepsilon}{\varepsilon-1})$ be Schrödinger admissible and assume the linear operator $T\colon L_t^1L_x^2\to L_x^2$, defined as

$$T(F)=\int_{\mathbb{R}}\ e^{-is\Delta}F(s,.)ds.$$

Its ad joint $T^*\colon L_x^2\to L_t^\infty L_x^2$ is the Schrödinger propagator as

$$T^*(u)=e^{i(s+\varepsilon)\Delta}u\,.$$

We can obtain the diagonal un truncated estimates by applying Minkowski's inequality, using (2.4) and $\frac{1}{\varepsilon} = \frac{1}{2} - \frac{4-\alpha}{4}$, *such that*

$0 < \alpha < 4 \,, 1 < \varepsilon \ as$

$$\left\| \int_{\mathbb{R}} e^{i(\varepsilon)\Delta} F(s,.)ds \right\|_{L^{\varepsilon}_{s+\varepsilon} L_x^{\frac{2\varepsilon}{\varepsilon-1}}(\mathbb{R}\times\mathbb{R}^4)} \leq \left\| \int_{\mathbb{R}} \left\| e^{i(\varepsilon)\Delta} F(s,.) \right\|_{L_x^{\frac{2\varepsilon}{\varepsilon-1}}(\mathbb{R}^4)} ds \right\|_{L^{\varepsilon}_{s+\varepsilon}(\mathbb{R})}$$

$$\lesssim \left\| \|F\|_{L_x^{\frac{2\varepsilon}{\varepsilon-1}'}(\mathbb{R}^4)} * \frac{1}{|s+\varepsilon|^{\frac{2}{\varepsilon}}} \right\|_{L^{\varepsilon}_{s+\varepsilon}(\mathbb{R})} \lesssim \|F\|_{L_{s+\varepsilon}^{\frac{\varepsilon}{\varepsilon-1}} L_x^{\frac{2\varepsilon}{3\varepsilon-1}}(\mathbb{R}\times\mathbb{R}^4)} .$$

Whenever, are such that $2 \leq \varepsilon$, and for any Schwartz function $F \in \mathcal{Y}(\mathbb{R}\times\mathbb{R}^4)$. Thus, using Lemma.1.1, we obtain the homogeneous Strichartz estimates (2.5) and the conformable dual homogeneous Strichartz estimates (2.6). Applied Lemma.1.1-(3) to the former two estimates yields the non-diagonal untruncate estimates:

$$\left\| \int_{\mathbb{R}} e^{i(\varepsilon)\Delta} F(s,.)ds \right\|_{L^{\varepsilon}_{s+\varepsilon} L_x^{\frac{2\varepsilon}{\varepsilon-1}}(\mathbb{R}\times\mathbb{R}^4)} \leq \|F\|_{L_{s+\varepsilon}^{\frac{\varepsilon}{\varepsilon-1}} L_x^{\frac{2\varepsilon}{3\varepsilon-1}}(\mathbb{R}\times\mathbb{R}^4)} .$$

We obtain by un truncated diagonal estimates the diagonal ones for the truncated operator, indicating that

$$\left\| \int_{-\infty}^{t} e^{i(\varepsilon)\Delta} F(s,.)ds \right\|_{L^{\varepsilon}_{s+\varepsilon} L_x^{\frac{2\varepsilon}{\varepsilon-1}}(\mathbb{R}\times\mathbb{R}^4)} \leq \left\| \int_{-\infty}^{t} \left\| e^{i\varepsilon\Delta} F(s,.) \right\|_{L_x^{\frac{2\varepsilon}{\varepsilon-1}}(\mathbb{R}^4)} ds \right\|_{L^{\varepsilon}_{s+\varepsilon}(\mathbb{R})}$$

$$\leq \left\| \int_{\mathbb{R}} \left\| e^{i\varepsilon\Delta} F(s,.) \right\|_{L_x^{\frac{2\varepsilon}{\varepsilon-1}}(\mathbb{R}^4)} ds \right\|_{L^{\varepsilon}_{s+\varepsilon}(\mathbb{R})} \lesssim \|F\|_{L_{s+\varepsilon}^{\frac{\varepsilon}{\varepsilon-1}} L_x^{\frac{2\varepsilon}{3\varepsilon-1}}(\mathbb{R}\times\mathbb{R}^4)} .$$

Furthermore, where $X = L_{s+\varepsilon}^{\frac{\varepsilon}{\varepsilon-1}} L_x^{\frac{2\varepsilon}{3\varepsilon-1}}$ and the truncated operator

$(T^{*}T)_R f(s+\varepsilon) = \int_0^{s+\varepsilon} e^{i\varepsilon\Delta} F(s,.)ds$, we can obtain

$$\left\| \int_{0}^{s+\varepsilon} e^{i\varepsilon\Delta} F(s,.)ds \right\|_{L^{\infty}_{s+\varepsilon} L^{2}_{x}(\mathbb{R}\times\mathbb{R}^{4})} \lesssim \|F\|_{L_{s+\varepsilon}^{\frac{\varepsilon}{\varepsilon-1}} L_{x}^{\frac{2\varepsilon}{3\varepsilon-1}}(\mathbb{R}\times\mathbb{R}^{4})}$$

For all admissible pairs $(\varepsilon, \frac{2\varepsilon}{\varepsilon-1})$.Then, by complex interpolation between this estimate and the diagonal truncated those mentioned above we get the non-diagonal truncate estimates (2.7), for any Schrödinger pair admissible $(\varepsilon, \frac{2\varepsilon}{\varepsilon-1})$, $\left(\tilde{\varepsilon}, \frac{\widetilde{2\varepsilon}}{\varepsilon-1}\right)$.

Theorem.2.3.

Let $2 \leq \varepsilon_j$, $2 \leq, \varepsilon_k$, such that $\varepsilon_j \leq \varepsilon_k$, and $d = 4$. Suppose the same for $\breve{\varepsilon}_j, \breve{\varepsilon}_k$.Then we have the homogeneous Strichartz estimates

$$\left\| e^{i(s+\varepsilon)\Delta} u_0 \right\|_{W(L^{\varepsilon_j}, L^{\varepsilon_k})_t W(L^{\frac{2\varepsilon_j}{\varepsilon_j-1}}, L^{\frac{2\varepsilon_k}{\varepsilon_k-1}})_x} \lesssim \|u_0\|_{L^2_x} \tag{2.8}$$

the dual homogeneous Strichartz estimates

$$\left\| \int e^{-is\Delta} F(s,.)ds \right\|_{L^2} \lesssim \|F\|_{W(L^{\frac{\widetilde{\varepsilon_j}}{\varepsilon_j-1}}, L^{\frac{\widetilde{\varepsilon_k}}{\varepsilon_k-1}})_t W(L^{\frac{\widetilde{2\varepsilon_j}}{\varepsilon_j+1}}, L^{\frac{\widetilde{2\varepsilon_k}}{\varepsilon_k-1}})_x} \tag{2.9}$$

and the retarded Strichartz estimates

$$\begin{aligned} \left\| \int_{\varepsilon>0} e^{i\varepsilon\Delta} F(s,.)ds \right\|_{W(L^{\varepsilon_j}, L^{\varepsilon_k})_t W(L^{\frac{2\varepsilon_j}{\varepsilon_j-1}}, L^{\frac{2\varepsilon_k}{\varepsilon_k-1}})_x} \\ \lesssim \|F\|_{W(L^{\frac{\widetilde{\varepsilon_j}}{\varepsilon_j-1}}, L^{\frac{\widetilde{\varepsilon_k}}{\varepsilon_k-1}})_t W(L^{\frac{\widetilde{2\varepsilon_j}}{\varepsilon_j+1}}, L^{\frac{\widetilde{2\varepsilon_k}}{\varepsilon_k-1}})_x} \end{aligned} \tag{2.10}$$

This result is obtained by first establishing the estimates for the case $\left(\varepsilon_j, \frac{2\varepsilon_j}{\varepsilon_j-1}\right) = (\infty, 2)$, $\left(\breve{\varepsilon}_j, \frac{\widetilde{2\varepsilon_j}}{\varepsilon_j-1}\right) = (\infty, 2)$, and then by complex interpolation with the homogeneous Strichartz estimates (2.5).

Figure1 explains the extent of exponents for the homogeneous estimates when $\mathrm{d} = 4$ Note that, if $\varepsilon_j \leq \varepsilon_k$, these estimates follow directly from the inclusion relations of Wiener amalgam spaces and $\left\| e^{i(s+\varepsilon)\Delta} u_0 \right\|_{W(L^{\varepsilon}, L^{\varepsilon})_{s+\varepsilon} W(L^{\frac{2\varepsilon}{\varepsilon-1}}, L^{\frac{2\varepsilon}{\varepsilon-1}})_x} \lesssim \|u_0\|_{L^2_x}$. Thus, the cause consists in the cases $\varepsilon_j > \varepsilon_k$. Since there are no connections between the pairs$(\varepsilon_j, \frac{2\varepsilon_j}{\varepsilon_j - 1})$ *and* $(\varepsilon_k, \frac{2\varepsilon_k}{\varepsilon_k - 1})$ other than $\varepsilon_j \leq \varepsilon_k$, through these estimates found that, the analysis of the local regularity of the Schrödinger propagator is quite independent of its decay at infinity.

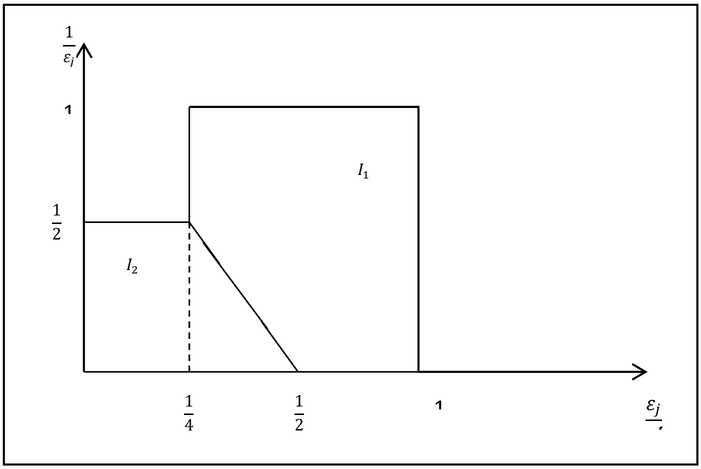

Figure1: $\boldsymbol{when\ d = 4, (2.7) holds for\ all\ pairs}\ \left(\frac{1}{\varepsilon_j}, \frac{\varepsilon_j - 1}{2\varepsilon_j}\right) \in \boldsymbol{I_1}, \left(\frac{1}{\varepsilon_k}, \frac{\varepsilon_k - 1}{2\varepsilon_k}\right) \in \boldsymbol{I_2\ with\ \varepsilon_k \leq \varepsilon_j}$

2.1 Strichartz estimates in the radial case:

In this subsection we use Christ-Kiselev lemma in [Christ.M, Kiselev. A, (2001)] to conclude the retarded linear estimates. We begin by prove a duality property for radial function.

Lemma.2.4.

Suppose that $p = 2, 1 = \frac{1}{p} + \frac{1}{p'}$, $f_i \in L^2(\mathbb{R}^4)$ and $\mathrm{f_i}$ is radial. Then

$$\sum_{i=1}^{r} \|f_i\|_{L^2(\mathbb{R}^4)} = sup \left\{ \begin{array}{c} \left| \int_{\mathbb{R}^4} \sum_{i=1}^{r} f_i(x) g_i(x)\, dx \right| : g_i \in L^2(\mathbb{R}^4), \\ g_i\ is\ radial\ and \sum_{i=1}^{r} \|g_i\|_{L^2} \leq 1 \end{array} \right\} \tag{2.11}$$

Proof: it is clear that the right-hand side of (2.11) is less than $\sum_{i=1}^{r} \|f_i\|_{L^2(\mathbb{R}^4)}$

thus it suffices to show $\sum_{i=1}^{r}\|f_i\|_{L^2(\mathbb{R}^4)}$ less than the right-hand side of (2.11)By duality, we have

$$\sum_{i=1}^{r}\|f_i\|_{L^2(\mathbb{R}^4)} = \sup_{g_i\in L^2,\sum_{i=1}^{r}\|g_i\|_{L^2}=1}\left|\int_{\mathbb{R}^4}\sum_{i=1}^{r} f_i(x)g_i(x)\,dx\right|$$

$$= \sup_{g_i\in L^2,\sum_{i=1}^{r}\|g_i\|_{L^2}=1}\left|\int_0^\infty\int_{S^3}\sum_{i=1}^{r} f_i(t)g_i(tx')t^3\,dtd\sigma(x')\right|$$

$$= \sup_{g_i\in L^2,\sum_{i=1}^{r}\|g_i\|_{L^2}=1}\left|\int_{\mathbb{R}^4}\sum_{i=1}^{r} f_i(x)\tilde{g}_i(x)\,dx\right|.$$

Where we put $\tilde{g}(x) = \frac{1}{|S^3|}\int_{S^3} g(|x|x')d\sigma(x')$. We see from Holder's inequality that g̃ radial and $\sum_{i=1}^{r}\ \|\tilde{g}_i\|_{L^2} \le 1$, then we get $\sum_{i=1}^{r}\|f_i\|_{L^2(\mathbb{R}^4)}$ less than the right-hand side of (2.11)as desirable.

Lemma.2.5.

Assume $1 \le \varepsilon, k \in \mathbb{Z}$, *if for all* $u_0 \in L^2(\mathbb{R}^4)$ *and* u_0 is radial we have

$$\left\|S_\varphi(s+\varepsilon)[P_k f(s+\varepsilon,s)](x)ds\right\|_{L^2(\mathbb{R}^4)} \lesssim c(k)\|f\|_{L_{s+\varepsilon}^{\frac{\varepsilon}{\varepsilon-1}}L_x^{\frac{2\varepsilon}{3\varepsilon-1}}}\ .$$

To prove see [Smith. H. F, Sogge. C. D, (2000)]

Lemma.2.6

Assume1 $\le \varepsilon_j, \varepsilon_k \le \infty, where \left(\varepsilon_j, \frac{2\varepsilon_j}{\varepsilon_j-1}\right), \left(\varepsilon_k, \frac{2\varepsilon_k}{\varepsilon_k-1}\right)$ are Schrödinger admissible $with\ \varepsilon_j > \varepsilon_k$If $for\ all f \in L_{s+\varepsilon}^{\varepsilon_k}L_x^{\frac{2\varepsilon_k}{\varepsilon_k-1}}$Spherically symmetric in space

$$\left\|\int_{\mathbb{R}} S_\varphi(\varepsilon)\big(P_k f(s)\big)(x)ds\right\|_{L_{s+\varepsilon}^{\varepsilon_j}L_x^{\frac{2\varepsilon_j}{\varepsilon_j-1}}} \lesssim c(k)\|f\|_{L_{s+\varepsilon}^{\varepsilon_k}L_x^{\frac{2\varepsilon_k}{\varepsilon_k-1}}}\ ,$$

then we have

$$\left\|\int_0^{s+\varepsilon} S_\varphi(\varepsilon)\big(P_k f(s)\big)(x)ds\right\|_{L_{s+\varepsilon}^{\varepsilon_j} L_x^{\frac{2\varepsilon_j}{\varepsilon_j-1}}} \lesssim c(k)\|f\|_{L_{s+\varepsilon}^{\varepsilon_k} L_x^{\frac{2\varepsilon_k}{\varepsilon_k-1}}}$$

holds with the same bound c(k), for all $f \in L_{s+\varepsilon}^{\varepsilon_k} L_x^{\frac{2\varepsilon_k}{\varepsilon_k-1}}$ spherically symmetric in space.

We note that, from Little wood-Paley square function theorem and Minkowski inequality we get if $\varepsilon \geq 2$ then

$$\|f\|_{L_{s+\varepsilon}^{\varepsilon} L_x^{\frac{2\varepsilon}{\varepsilon-1}}} \lesssim \left\| \|P_k f\|_{L_{s+\varepsilon}^{\varepsilon} L_x^{\frac{2\varepsilon}{\varepsilon-1}}} \right\|_{L_k^2}, \quad \left\| \|P_k f\|_{L_{s+\varepsilon}^{\frac{\varepsilon}{\varepsilon-1}} L_x^{\frac{2\varepsilon}{3\varepsilon-1}}} \right\|_{L_k^2} \lesssim \|f\|_{L_{s+\varepsilon}^{\frac{\varepsilon}{\varepsilon-1}} L_x^{\frac{2\varepsilon}{3\varepsilon-1}}} \quad (2.12)$$

Furthermore, consider (1.1) and if u_0 is radial then

$$\left\|e^{-i(s+\varepsilon)\Delta} P_k u_0\right\|_{L_{s+\varepsilon,x}^{\varepsilon}} \lesssim 2^{\left(2-\frac{6}{\varepsilon}\right)k}\|u_0\|_2 \quad (2.13)$$

Proposition.2.7

(Schrödinger Strichartz estimate). Assume $d = 4$ *and* u, u_0, F are spherically symmetric satisfy equation (1.1). If $\alpha \in \mathbb{R}$, $\left(\varepsilon, \frac{2\varepsilon}{\varepsilon-1}\right), (\delta, \frac{2\delta}{\delta-1})$ are both n-D radial Schrödinger-admissible, where $\varepsilon \neq 2,1$ or $\delta \neq 2,1$, and $\alpha, \delta, \varepsilon$ satisfy the "gap" condition

Then

$$\|u\|_{L_{s+\varepsilon}^{\varepsilon} L_x^{\frac{2\varepsilon}{\varepsilon-1}}} + \|u\|_{C(\mathbb{R};\dot{H}^s)} \lesssim \|u_0\|_{\dot{H}^s} + \|u\|_{L_{s+\varepsilon}^{\frac{\delta}{\delta-1}} L_x^{\frac{2\delta}{3\delta-1}}} \quad (2.14)$$

Proof: we first prove the case when $\mathrm{F} = 0$. Assume $\left(\varepsilon, \frac{2\varepsilon}{\varepsilon-1}\right)$, is n-D radial Schrödinger admissible, and by scaling, it suffices to prove

$$\left\|e^{-i(s+\varepsilon)\Delta} P_0 u_0\right\|_{L_{s+\varepsilon}^{\varepsilon} L_x^{\frac{2\varepsilon}{\varepsilon-1}}} \lesssim \|u_0\|_{L_x^2} \quad (2.15)$$

In view of the known results of Strichartz estimate (see [KEEL, M. AND TAO.T, (1998)], [Tao. T (2000)]), we see that (2.15) hold if

$$\frac{1}{2\varepsilon} + \frac{\varepsilon-1}{2\varepsilon} \leq 1 .$$

Now return to prove the case $F \neq 0$, $\left(\varepsilon, \frac{2\varepsilon}{\varepsilon-1}\right)$ and $(\delta, \frac{2\delta}{\delta-1})$ are both n-D radial Schrödinger admissible, $\delta \neq 2{,}1$ and satisfy the “gap” condition. By the already known estimates [KEEL, M. AND TAO.T ,(1998)] we implied the case $\alpha = 0$. $If\, \alpha \neq 0$, then by scaling it suffices to prove

$$\left\| \int_{0}^{s+\varepsilon} e^{-i(\varepsilon)\Delta} P_0 F(s) ds \right\|_{L^{\varepsilon}_{s+\varepsilon} L_x^{\frac{2\varepsilon}{\varepsilon-1}}} \lesssim \|F\|_{L_{s+\varepsilon}^{\frac{\delta}{\delta-1}} L_x^{\frac{2\delta}{3\delta-1}}} \quad .$$

Since either $\varepsilon > 2$ or $\delta > 2$, thus, in view of Christ-Kiselev lemma it suffices to prove

$$\left\| \int_{\mathbb{R}} e^{-i(\varepsilon)\Delta} P_0 F(s) ds \right\|_{L^{\varepsilon}_{s+\varepsilon} L_x^{\frac{2\varepsilon}{\varepsilon-1}}} \lesssim \|F\|_{L_{s+\varepsilon}^{\frac{\delta}{\delta-1}} L_x^{\frac{2\delta}{3\delta-1}}} \quad ,$$

which follows directly from Lemma. 2.5 and the non-retarded linear estimates.

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